1) skeleton + coskeleton functors

2) hypercovers

3) homotopy (co)limits

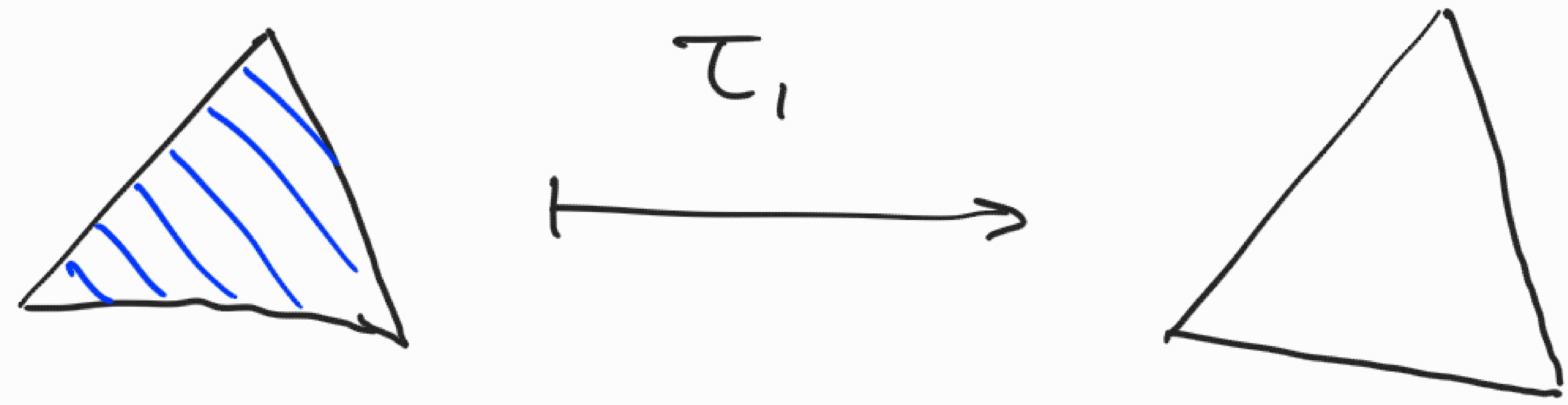
Étale Homotopy Study Group
Session 2 (5.10.20)

N. Sale

Skeletons + Coskeletons

defn: Δ/n is the *n-truncated ordinal category*. It is constructed from Δ by removing the objects $[m]$ for all $m > n$ (along with any morphisms to and from these). Denote by $S_n C$ the category of contravariant $\Delta/n \to C$.

prop: Given a category C closed under finite limits and colimits, as well as $n \in \mathbb{N}$, then there exists a truncation functor $\tau_n : SC \to S_n C$. Moreover this functor has both a left and right adjoint $\tau_n^L \dashv \tau_n \dashv \tau_n^R$.



defn: The <u>n-skeleton functor</u> $Sk_n : s\mathcal{C} \to s\mathcal{C}$ is defined to be the composition

$$s\mathcal{C} \xrightarrow{\tau_n} s_n\mathcal{C} \xrightarrow{\tau_n^L} s\mathcal{C}$$

The <u>n-coskeleton functor</u> $Cosk_n : s\mathcal{C} \to s\mathcal{C}$ is the composition

$$s\mathcal{C} \xrightarrow{\tau_n} s_n\mathcal{C} \xrightarrow{\tau_n^R} s\mathcal{C}$$

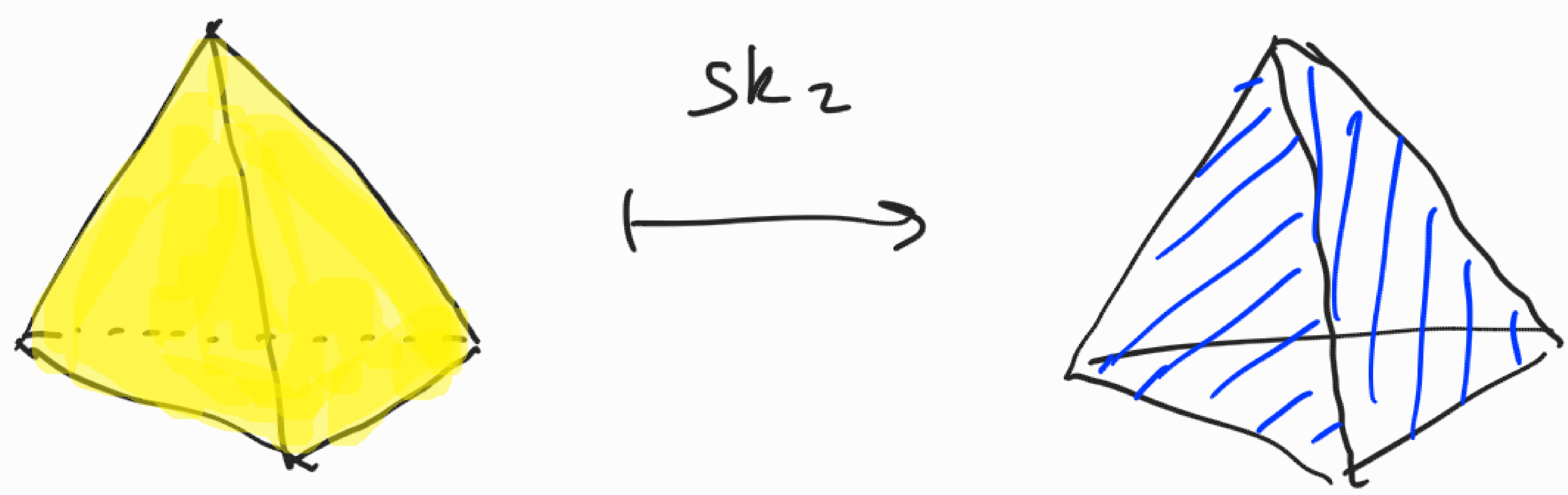
intuition

- Think of τ_n as a forgetful functor, forgetting some of the simplicial object's structure
- <u>Recall</u> : The left adjoint of forgetful functors are often called free functors.

They generate just enough of the forgotten Structure

e.g: Free : $\underline{\mathrm{Set}} \longrightarrow \underline{\mathrm{Grp}}$

- Hence sk_n first throws away all the m-Simplices for $m > n$, then fills in just enough degenerate simplices to make it a well-defined simplicial object again.



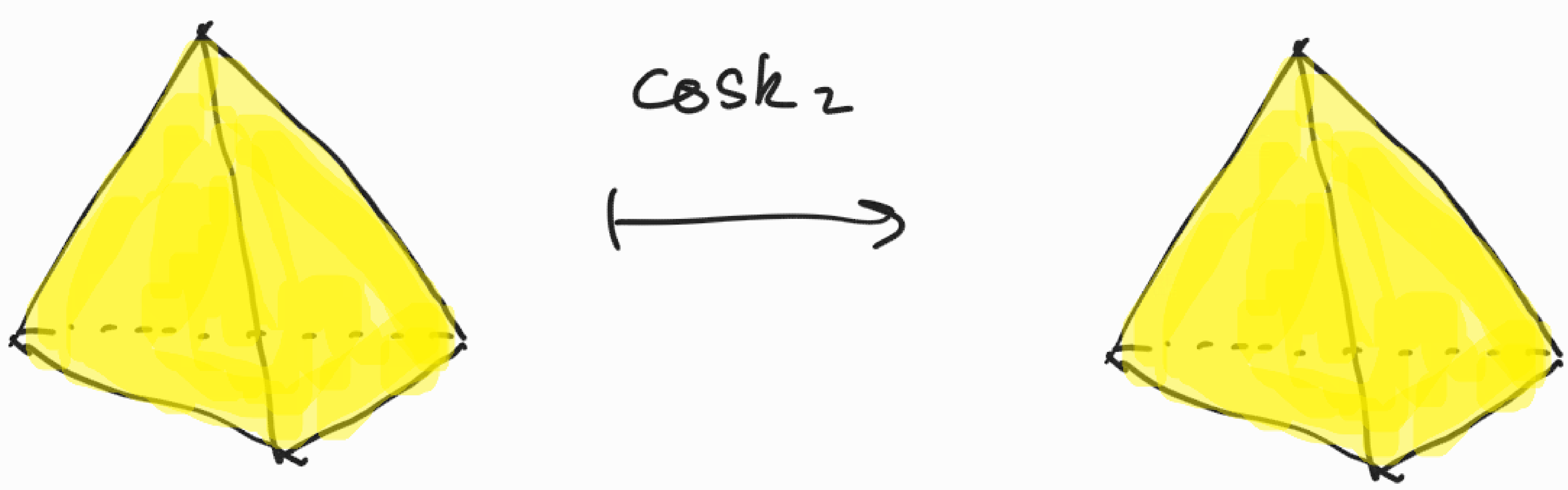
- Right adjoints of forgetful functors are often called cofree functors. They are a little overzealous in filling in missing structure

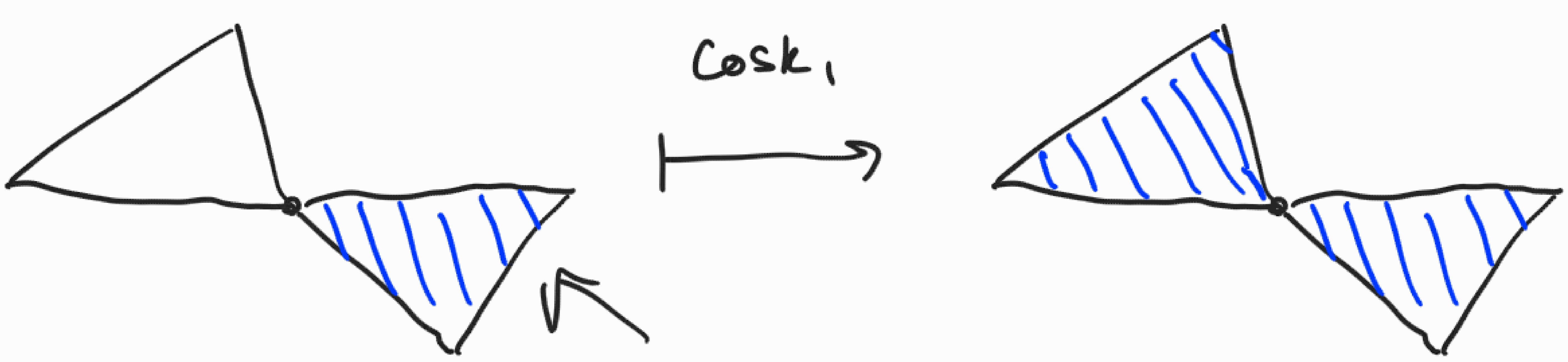
e.g. cofree : $\underline{\mathrm{Set}} \longrightarrow \underline{\mathrm{Cat}}$

produces a category whose objects

come from the set with morphisms $x \to y$ for every pair of objects x, y.

- Hence cosk_n does at first the same as sk_n, but will also reintroduce non-degenerate $(n+1)$-simplices if it can: i.e. if all the faces are non-degenerate.





(note that cosk_1 kills π_1)

Note that by the defn of adjoint functors we have:

$$\mathrm{Hom}(\mathrm{sk}_n A, B) \cong \mathrm{Hom}(\tau_n A, \tau_n B) \cong \mathrm{Hom}(A, \mathrm{cosk}_n B)$$

Setting $A = B$ yields maps

$Sk_n A \to A$ and $A \to Cosk_n A$

corresponding to $id : \tau_n A \to \tau_n B$.

prop: If X is from $\mathcal{H}_0$, then $Cosk_n X$ is characterised by a universal property:

(1) $\pi_m(Cosk_n X) = 0 \quad \forall m \geq n$

(2) For all maps $X \to Y$ s.t. $\pi_m Y = 0 \quad \forall m \geq n$ we have

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & \circlearrowleft & Y \\ & Cosk_n X \arrow[ur, dashed, "\exists"'] & \end{tikzcd}.$$

This universal property shows that we have maps $Cosk_{n+1} X \to Cosk_n X \quad \forall n$.

Its also the case that the maps

$X \to Cosk_n X$ are weak equivalences.

In fact we can draw a commutative diagram:

$$\begin{tikzcd}
& \vdots \arrow[d] \\
& \mathrm{Cosk}_2 X \arrow[d] \\
& \mathrm{Cosk}_1 X \arrow[d] \\
X \arrow[r] \arrow[ur] \arrow[uur] & \mathrm{Cosk}_0 X
\end{tikzcd}$$

and this kind of diagram, when we have the property that each of the maps from X is a weak equiv., is called a Postnikov tower.

Essentially a way to decompose a space w.r.t its different homotopy groups. In fact, when the vertical maps are fibrations, the fibres are $K(\pi_n X, n)$ spaces.

Eilenberg-MacLane space

Hypercovers

Given X a CW complex and $\{U_\alpha\}$ an open cover, we can think of the cover as a map

$$\mathcal{U} \longrightarrow X$$

with $\mathcal{U} = \coprod_\alpha U_\alpha$.

In this case, the pairwise intersections $U_\alpha \cap U_\beta$ can be seen as the fibre product $U_\alpha \times_X U_\beta$, so that $\mathcal{U} \times_X \mathcal{U} = \coprod_{\alpha,\beta} U_\alpha \cap U_\beta$. Similarly

$$\mathcal{U} \times_X \mathcal{U} \times_X \mathcal{U} = \coprod_{\alpha,\beta,\gamma} U_\alpha \cap U_\beta \cap U_\gamma$$

and so on.

<u>defn</u>: Given a cover $\mathcal{U} \to X$, we construct the <u>Čech nerve</u> $\pi_0(\mathcal{U})$

of the cover as the sSet with

$$\pi_0(\mathcal{U})_n := \underset{\substack{\uparrow \\ \text{connected} \\ \text{components} \\ \text{of}}}{\pi_0} (\underbrace{\mathcal{U} \times_X \cdots \times_X \mathcal{U}}_{n+1 \text{ times}})$$

with face maps coming from the 'projections' $\mathcal{U} \times_X \mathcal{U} \rightrightarrows \mathcal{U}$

$$\mathcal{U} \times_X \mathcal{U} \times_X \mathcal{U} \to \mathcal{U}$$

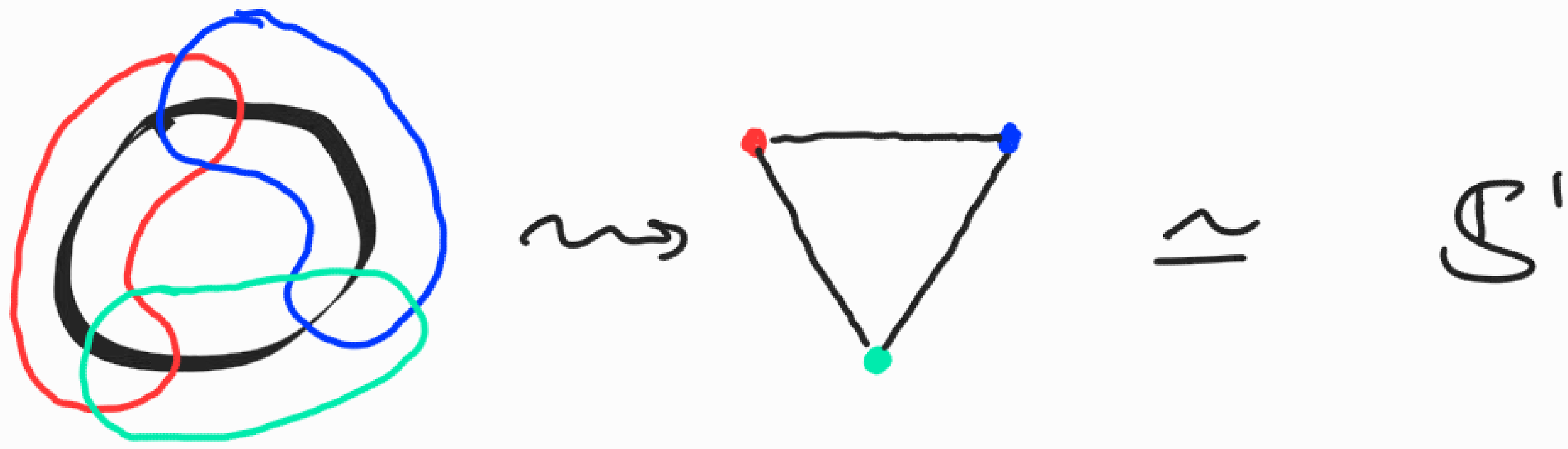
etc...

and degeneracy maps from diagonal embeddings.

<u>defn</u>: A cover $\bigcup_\alpha U_\alpha = X$ is <u>good</u> if every intersection of the U_α is either $\emptyset$ or a disjoint union of contractible subsets.

$\simeq *$

<u>thm</u>: (sSet Nerve Theorem) If $\mathcal{U} \to X$ is a good cover, then $|\pi_0(\mathcal{U})|$ is weakly equivalent to X.



This is great, but étale covers are generally not good. So we need a more general construction than the Čech nerve.

<u>defn</u>: A simplicial object $\mathcal{U}_\bullet$ is a <u>hypercover</u> of X if:

(1) $\mathcal{U}_0 \to X$ is a cover

(2) $\mathcal{U}_{n+1} \to \mathrm{Cosk}_n(\mathcal{U}_\bullet)_{n+1}$ is a cover.

<u>intuition</u>

- we cover X with $\mathcal{U}_0$
- we cover $\mathcal{U}_0 \times_X \mathcal{U}_0$ with a potentially new cover $\mathcal{U}_1$

- we cover $\mathcal{U}_1 \times_X \mathcal{U}_1 \times_X \mathcal{U}_1$ with a potentially new cover $\mathcal{U}_2$
- etc ...

<u>defn</u>: Let $\mathcal{C}$ be a cat. with finite dir sums. If $K \in \text{ob}\,\mathcal{C}$ and S is a finite set, then <u>$K \otimes S$</u> denotes the direct sum of copies of K indexed by the elements of S.

Let $(K_\bullet)$ be in $S\mathcal{C}$. Define <u>$(K_\bullet) \otimes \Delta[1]$</u> by $((K_\bullet) \otimes \Delta[1])_n := \underline{K_n \otimes \Delta[1]_n}$.

Let e_0, e_1 be the inclusions

$$(K_\bullet) \longrightarrow (K_\bullet) \otimes \Delta[1]$$

corresponding to the two inclusions

$$\Delta[0] \longrightarrow \Delta[1].$$

defn: Two maps $f, g : K_\bullet \to L_\bullet$ are **strictly homotopic** if $\exists\, h : K_\bullet \otimes \Delta[1] \to L_\bullet$ s.t. $f = h e_0$ and $g = h e_1$. Two maps are **homotopic** if linked by a chain of strict homotopies.

defn: The homotopy category of hypercoverings $HC(X_{ét})$ is the category of étale hypercoverings of X and whose morphisms are homotopy classes of morphisms of $SX_{ét}$.

prop: $HC(X_{ét})$ is cofiltering.

Thm: $H^\wedge_{ét}(X) = \varinjlim H^\wedge(\mathcal{U}_\bullet)$.

Homotopy limits + Colimits

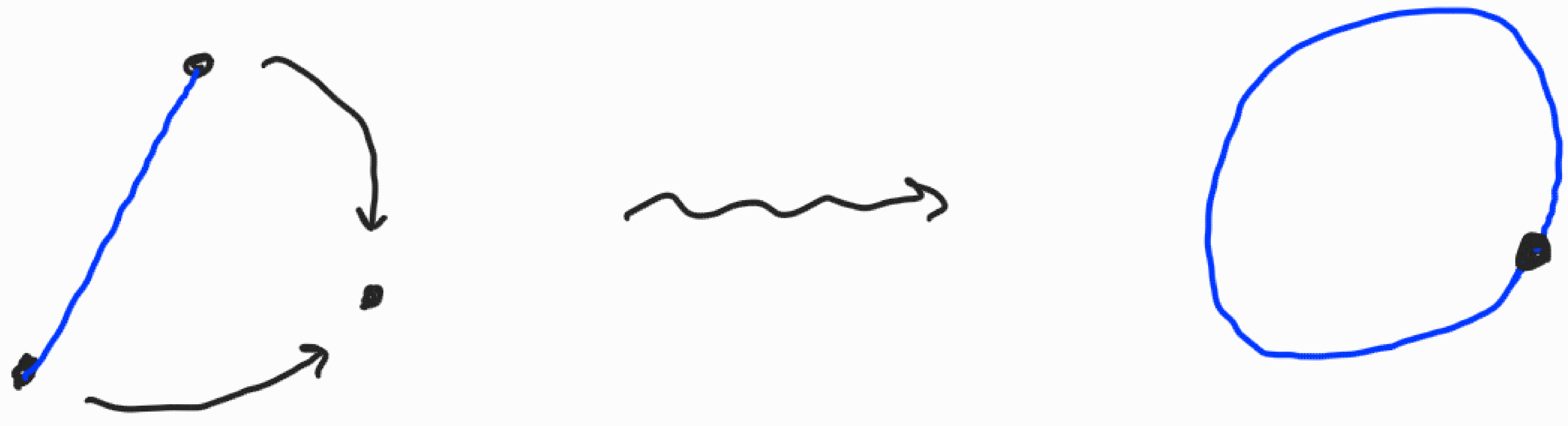
Motivation

- when we replace an object in a diagram $F : I \to \mathcal{C}$ by something weakly equivalent, the resulting (co)limit is not always weakly equivalent to $\varprojlim F$ ($\varinjlim F$).

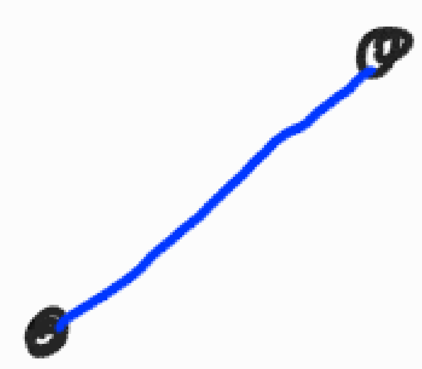
This is not very helpful for homotopy theory.

- e.g.: The pushout $D^n \sqcup_{S^{n-1}} *$ is S^n.

$n=1$



$n = 2$ 

Now  $\simeq \; * \; \simeq$ 

but $* \sqcup * = *$ (not a sphere).

generally

- Let $\mathcal{C}^I$ be the category of diagrams $I \to \mathcal{C}$ of a category $\mathcal{C}$.

- There is a diagonal functor

$$\Delta_0 : \mathcal{C} \longrightarrow \mathcal{C}^I$$

that sends X to the diagram which has X and $id : X \to X$ everywhere

The usual limit is right adjoint to Δ_0.

- If we replace Δ_0 by a functor

$$\Delta : \mathcal{C} \longrightarrow \mathcal{C}^I$$

which sends X to the diagram

$$i \longmapsto X \times |N(I/i)|$$

geometric realisation

nerve

Slice category of morphisms $j \to i$ $\forall j \in I$.

then holim is right adjoint to Δ.

- Similarly colim is left adjoint to Δ_0.

Now hocolim is left adjoint to the functor $\Delta : \mathcal{C} \to \mathcal{C}^I$

$$\Delta(X)(i) = \mathrm{Hom}_{\mathcal{C}}(|N(I^{op}/i)|, X).$$

CW case

- When considering the (co)limit of a finite diagram of CW complexes we have a nice way to understand holims and hocolims.

- Instead of cones (N, Ψ) on the diagram $F: I \to C$ where we have

$$\begin{tikzcd} & N \arrow[dl, "\Psi(X)"'] \arrow[dr, "\Psi(Y)"] & \\ F(X) \arrow[rr, "F(f)"] & & F(Y) \end{tikzcd}$$

commuting $\forall f: X \to Y$

we instead consider a homotopy coherent cone where instead we have that

$$F(f) \circ \Psi(X) \overset{h}{\sim} \Psi(Y)$$

and for any two such homotopies h, g we have that

$$h \overset{h'}{\sim} g$$

and for any two such h', g' we have that $h' \overset{h''}{\sim} g'$

and so on ...

- Then the holim is the homotopy coherent cone with the universal property that all other such cones factor through it.

- Similarly for hocolim and cocones.

homotopy pushout of $A \leftarrow B \rightarrow C$

is $A \sqcup_1 B \times [0,1] \sqcup_0 B \sqcup_1 B \times [0,1] \sqcup_0 C$

Take our previous example:

$D^2 \leftarrow S^1 \longrightarrow *$

The homotopy pushout is now

replacing D^2 with $*$ yields