Persistent Homology of $\mathbb{Z}_2$ Gauge Theories

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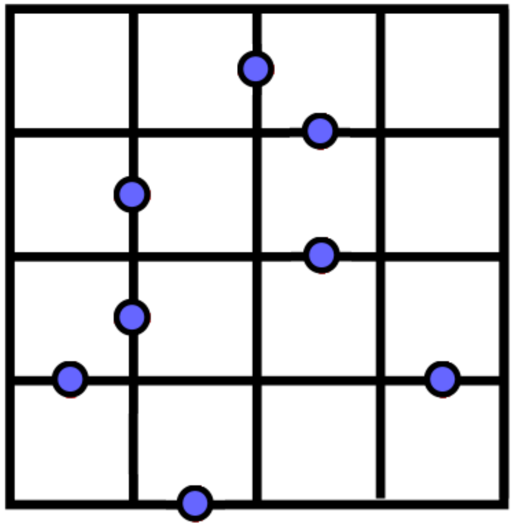
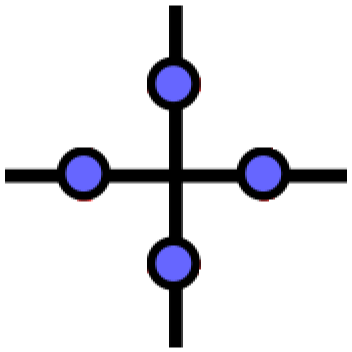
Topologically ordered phases of matter display a number of unique characteristics, including ground states that can be interpreted as patterns of closed strings. In this paper, we consider the problem of detecting and distinguishing closed strings in Ising spin configurations sampled from the classical $\mathbb{Z}_2$ gauge theory. We address this using the framework of persistent homology, which computes the size and frequency of general loop structures in spin configurations via the formation of geometric complexes. Implemented numerically on finite-size lattices, we show that the first Betti number of the Vietoris-Rips complexes achieves a high density at low temperatures in the $\mathbb{Z}_2$ gauge theory. In addition, it displays a clear signal at the finite-temperature deconfinement transition of the three-dimensional theory. We argue that persistent homology should be capable of interpreting prominent loop structures that occur in a variety of systems, making it an useful tool in theoretical and experimental searches for topological order.

$\mathbb{Z}_2$ Gauge Theory

- Square lattice (here 2D or 3D)
- Binary variable in {-1, 1} on each link
- Gauge transformations flip the variables on edges around 1 vertex

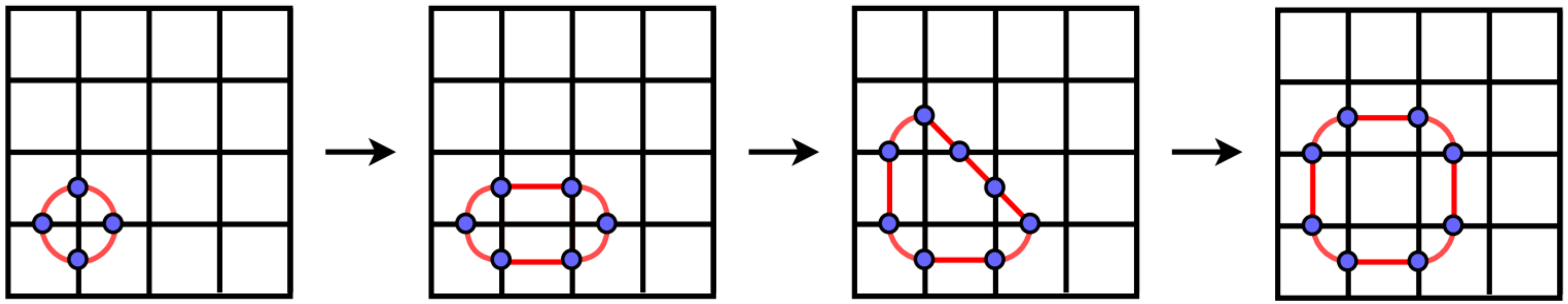




$$H = -K \sum_{\Box} P_{\Box}$$

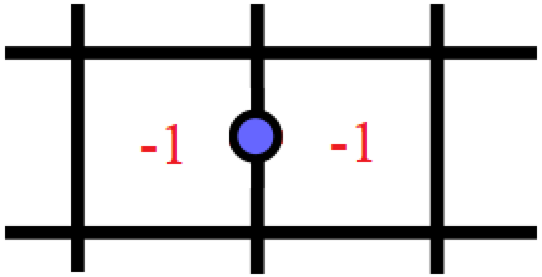
Ground States and Loops

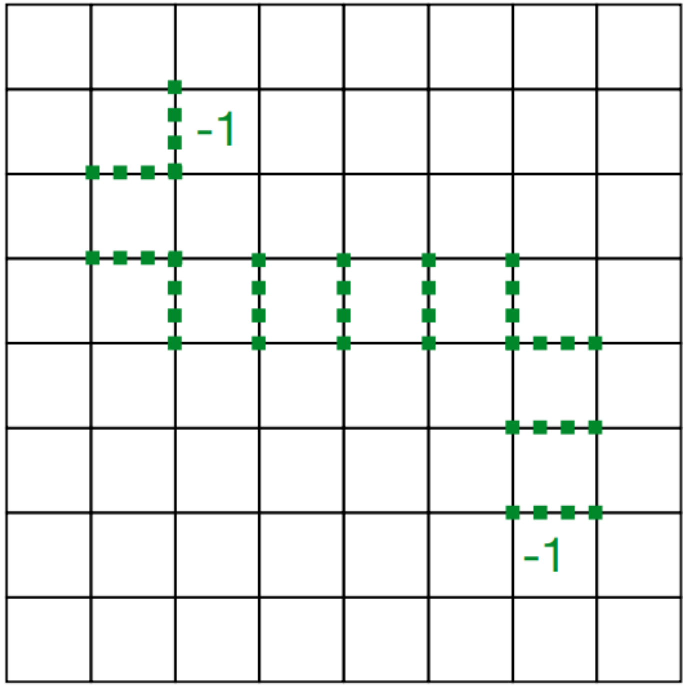
- Minimal energy obtained if all plaquettes have an even number of occupied links
- In this case, occupied links form loops (in the dual lattice)
- Achieved via gauge transformations of unoccupied configuration



Visons

- Excitations come from an odd number of occupied links
- These form a pair of visons
- Gauge transformations may pull these apart to form an open string of occupied links





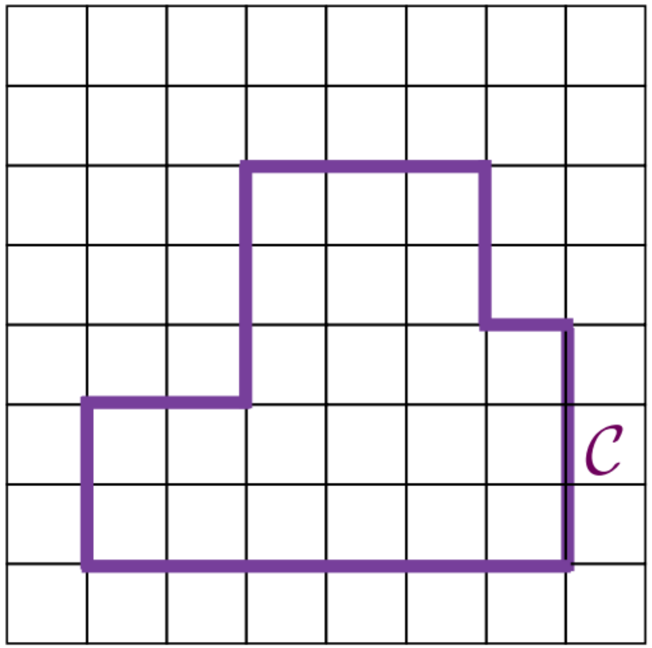
Phase Transition

- In 2D there is no phase transition
- In 3D there is a deconfinement transition
  - Low temperature (high K)

$$\left\langle \prod_{\ell \in C} \sigma(\ell) \right\rangle = \mathrm{e}^{-h(K)L}$$

  - High temperature (low K)

$$\left\langle \prod_{\ell \in C} \sigma(\ell) \right\rangle = \mathrm{e}^{-f(K)A}$$

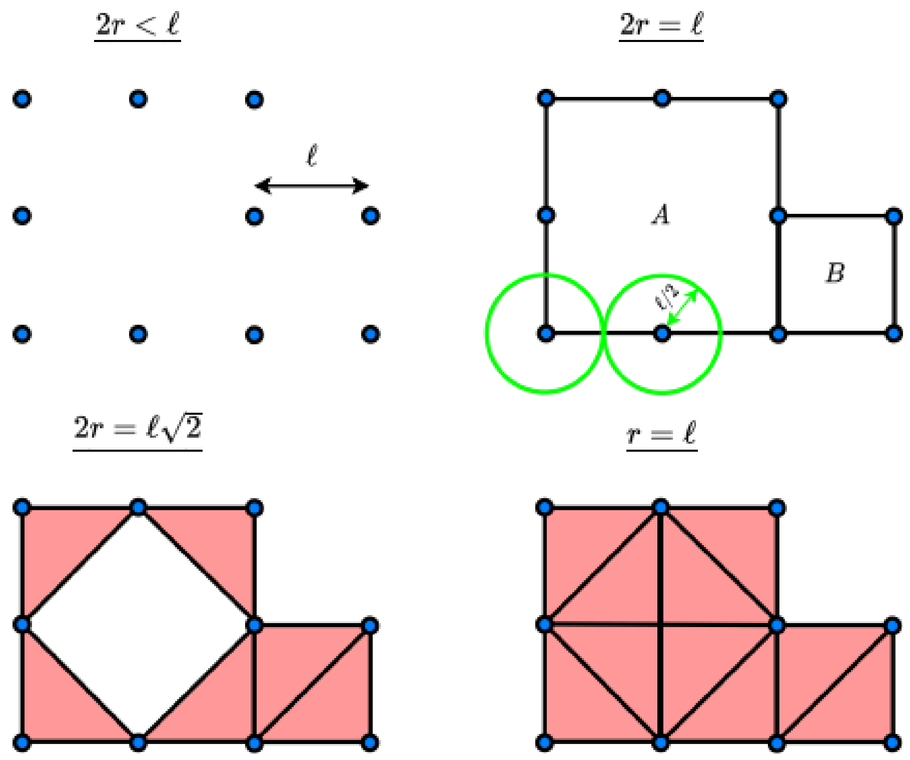


(Persistent) Homology

- Given point cloud of occupied links
- β_1 of the Vietoris-Rips complex with

$$\ell/\sqrt{2} \leq r < \ell$$

  captures all loops of occupied sites except those around a single vertex
- Persistence can tell us about the size of the loops



Experimental Setup

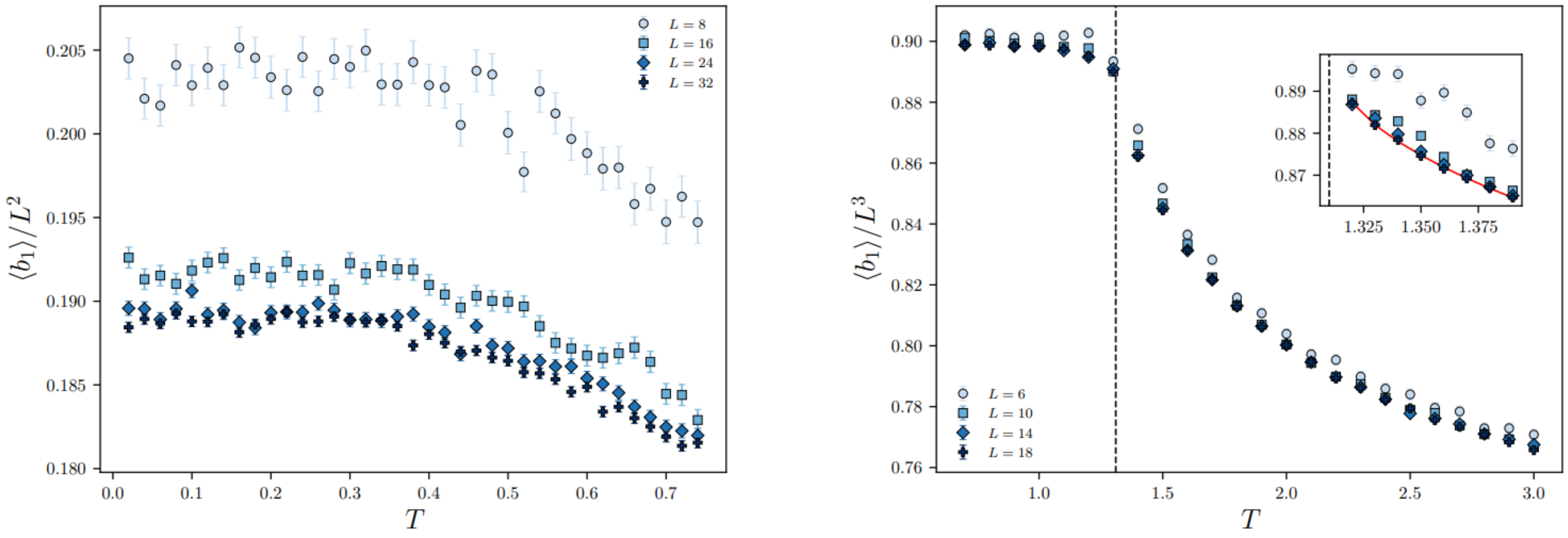
- Over a range of temperatures (i.e. values of K) and lattice sizes
- Obtain 2000 configurations via Metropolis sampling
  - Cluster updates with random gauge transformations
  - Spin flips (accepted with probability based on energy change)
  - Periodic boundary conditions
- Point cloud of occupied links with distance between points

$$d(\boldsymbol{x}, \boldsymbol{y}) = \sqrt{\sum_{\alpha=1}^{D} \min\left[y_\alpha - x_\alpha, (x_\alpha - \ell_1) + (\ell_2 - y_\alpha)\right]^2}$$

- Record the average β_1 of the configs for each temperature

Results

- Higher temperatures -> more visons -> lower β_1
- Larger lattice size -> larger loops -> less loops
- Sharp indicator of phase transition in 3D



Discussion

- Approach not gauge-invariant
  - Generalizable to systems where local symmetry not known explicitly
  - Would this be an issue for larger gauge groups?
- Scaling
  - Notes a power law scaling for β_1 at temperatures above critical point
  - Straightforward step to produce estimate of critical temperature